Some Asymptotic Results in Turbulent Diffusion

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The frozen-in case of turbulent diffusion requires for certain stochastic monotonic functions, for which a central limit theorem holds.

In a recent paper [1] it was shown how the turbulent velocity field can be generated together with the requested transition probabilities for a test partile by means of a suitable random field $S(t, x)$, if

$$S(t, x) = \text{const} = S(t = 0, x_0) \tag{0.1}$$

defines the trajectories for any individual realisation of the velocity field. This implies that

$$dx/dt = v(t, x) \tag{0.2}$$

is the equation of motion along the trajectory, with

$$v(t, x) = - S_t/S_x \tag{0.3}$$

at least in the 1-dimensional case, to which we confine ourselves in the following.

Compared with conventional Brownian motion, the perhaps most opposite situation would be represented by what we call "frozen-in turbulence". In this case, the velocity field is time independent and therefore any diffusion of test particles stems essentially from the spatial randomness of the fluid motion. We have already treated several examples in our previous paper. However, since the determination of the diffusion requires the solution of a certain Fokker Planck equation [1] one of the most interesting cases was not fully discussed.

Meanwhile this has been done by F. Pohl by a computer [2]. The results suggest a certain generalisation which we offer in the following.

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1. The Frozen-in Case

Let

$$S(t, x) = \varrho(x) - t. \tag{1.1}$$

For the corresponding velocity field we obtain

$$v(x) = 1/\varrho'(x) \tag{1.2}$$

independent of time (= frozen-in). Let us assume $\varrho(x)$ to be a monotonically growing stochastic function, which hence allows the representation

$$\varrho(x) = \int_0^x \frac{dx'}{\chi^2(u(x'))}, \tag{1.3}$$

where $\chi(u)$ is assumed to be a real function of u. *Furthermore, for $u = u(x)$ we assume a stationary Ornstein-Uhlenbeck process in x.* Hence $v(x)$ represents some kind of turbulent wind from the left-hand side:

$$v(x) = \chi^2(u(x)). \tag{1.4}$$

From this relation the probability distributions of the velocity field v can be deduced from those of the original Ornstein-Uhlenbeck-process. (Of course, χ should be a measurable map.)

Our main task would be to find the transition probabilities for the particles starting at, for example, $x = 0$ initially (at $t = 0$). Since our fluid turbulence is homogeneous (as is the underlying Ornstein-Uhlenbeck process), we expect

$$\text{Prob}\{t: x_0 \to x\} = \text{Prob}\{t = o: 0 \to x - x_0\} \tag{1.5}$$

to give a complete answer even when treating this somewhat reduced question. (This is connected with

the "family" problem, compare Wentzell [4].) It turns out that we are primarily obliged to look for the probability

$p(x: \varrho, u)\, \mathrm{d}\varrho\, \mathrm{d}u$

$\quad = \mathrm{Prob}\,\{\varrho \leq \varrho(x) \leq \varrho + \mathrm{d}\varrho,\, u \leq u(x) < u + \mathrm{d}u\},$

for which the following Fokker-Planck eqution holds [1]:

$$\frac{\partial P}{\partial x} + \frac{1}{\chi^2(u)}\frac{\partial P}{\partial \varrho} = \frac{1}{2}\frac{\partial^2 P}{\partial u^2} + \beta \frac{\partial}{\partial u}(uP). \quad (1.6)$$

The corresponding initial conditions for a transition probability are

$$P(x = 0;\varrho, u) = \delta(\varrho)\frac{e^{-u^2/2}}{\sqrt{2\pi}}, \quad (1.7)$$

where we have assumed for simplicity the Ornstein-Uhlenbeck process in a normalized form ($\langle u^2\rangle = 1$).

Hence we are left with solving the above differential equation for P and afterwards considering

$$q(t: x) = \int \frac{1}{\chi^2(u)}\, P(x: t, u)\, \mathrm{d}u \quad (1.8)$$

in order to obtain the transition probability of interest [1]. One of the most ineresting examples,

$$\chi^2(u) = \varepsilon + u^2, \quad (1.9)$$

which we already mentioned in the previous paper, has meanwhile been tackled. It turns out that the result for large values of x can be understood directly and it is not confined to the above special choice (1.9) of $\chi(u)$. Pohl [2] found that the probability distributions

$$P(x: \varrho) = \int P(x: \varrho, u)\, \mathrm{d}u \quad (1.10)$$

can be matched with excellent accuracy by a Gaussian, if $x \to \infty$. This behaviour can be understood by means of the central limit theorem. We want to show, why and how.

2. The Role of the Central Limit Theorem

Consider the definition of ϱ

$$\varrho(x) = \int_0^x \frac{\mathrm{d}x'}{\chi^2(u(x'))}. \quad (2.1)$$

Here $u(x)$ in an Ornstein-Uhlenbeck process obeying an Ito equation

$$\mathrm{d}u(x) = -\beta u\, \mathrm{d}x + \xi(x)\sqrt{\mathrm{d}x}, \quad (2.2)$$

where $\langle \xi\rangle = 0$ and

$$\langle \xi(x)\,\xi(x')\rangle = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise.} \end{cases} \quad (2.3)$$

Hence $\xi\sqrt{\mathrm{d}x}$ is the Wiener differential $\mathrm{d}w(x)$ (P.Levy [3]). From the above we find as a correlation length

$$\lambda_{\mathrm{corr}} = 1/\beta, \quad (2.4)$$

since the correlation is given by

$$\gamma = \langle u(x_1)\,u(x_2)\rangle = e^{-\beta|x_2-x_1|}. \quad (2.5)$$

This implies that the Ornstein-Uhlenbeck process becomes independent for large distances $|x_2 - x_1|$; in fact, its 2-point distribution reads as

$P_2(x_1: u_1;\, x_2: u_2)$

$$= \frac{1}{2\pi\sqrt{1-\gamma^2}}\exp\left\{-\frac{1}{2(1-\gamma^2)}(u_1^2 - 2\gamma u_1 u_2 + u_2^2)\right\}$$

$$\to \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{u_1^2}{2}\right\}\cdot\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{u_2^2}{2}\right\} \quad (2.6)$$

in the limiting case under consideration. Hence, in principle, we expect a similar behaviour for

$$v(x) = \chi^2(u((x)):$$

$P_2(x_1: v_1;\, x_2: v_2)$ should factorize as

$$P_1(x_1: v_1)\, P_1(x_2: v_2)\quad \text{for}\quad |x_2 - x_1| \to \infty. \quad (2.7)$$

It seems reasonable that the correlation length for the v-process would be of the same order of magnitude as for the underlying Ornstein-Uhlenbeck process. If so, the above-mentioned property for v similarly holds for $1/\chi^2(u(x))$ as well, and hence we may break the integral

$$\varrho(x) = \int_0^x \frac{\mathrm{d}x'}{\chi^2(u(x'))}$$

for larger values of x into a sum

$$\varrho(x) = \sum_{k=0}^{x/\lambda} \int_{k\lambda}^{(k+1)\lambda} \frac{\mathrm{d}x'}{\chi^2(u(x'))} = \sum I_k. \quad (2.8)$$

For any integer x/λ this formula is rigorous. We now introduce the assumption that the different terms in the sum are also uncorrelated:

$$\langle I_k \, I_{k+j} \rangle = 0 \tag{2.9}$$

and, furthermore, are independent. This may be just an approximate result, which at the moment sounds reasonable: It is based on the observation that a stochastic process ϱ with infinite correlation length cannot have a derivative $1/\chi^2$ with finite correlation. However, λ in (2.8) could be a few times larger than the λ of (2.4). Accepting the approximation for large values of x, we now end up with the fact that ϱ should have a Gaussian distribution:

$$P(x: \varrho) = \frac{\exp\left\{-\dfrac{(\varrho - U(x))^2}{2\,E^2(x)}\right\}}{\sqrt{2\pi\,E^2(x)}} \quad \text{for} \quad x \to \infty . \tag{2.10}$$

This result holds independently of the very special choice of $\chi^2(u)$, except that we should perhaps avoid divergences of ϱ. For example, χ should never vanish for any finite u. In the previous choice (1.9) ($\chi^2 = \varepsilon + u^2$) this was guaranteed by $\varepsilon > 0$.

In any case, let

$$\varepsilon = \text{Min}_u \, \chi^2(u) . \tag{2.11}$$

If $\varepsilon > 0$, we obtain the trivial estimate

$$\varrho = \int_0^x \frac{\mathrm{d}x'}{\chi^2(u(x'))} \le \frac{x}{\varepsilon} \tag{2.12}$$

and hence

$$P(x: \varrho) = 0 \tag{2.13}$$

outside the interval $0 \le \varrho < x/\varepsilon$. For any finite x this would be in contradiction with a Gaussian distribution, but the error will be negligible if

$$\eta = \text{Prob}\left\{|\varrho - U(x)| > U(x)\right\} \tag{2.14}$$

is small enough. Using the Chebychev inequality, we find

$$\eta \le \frac{E^2(x)}{U^2(x)} .$$

According to the subsequent result, the right-hand side behaves as

$$\frac{\Delta}{\mu^2} \cdot \frac{1}{x} \tag{2.15}$$

for large values of x, and hence the total error becomes negligible.

3. The Average Shift (the "Drag")

We are concerned with determining $U(x)$. This can be done rigorously by using the definition of ϱ

$$U(x) = \langle \varrho(x) \rangle = \int \varrho \, \mathrm{d}\varrho \, P(x: \varrho, u) \, \mathrm{d}u \tag{3.1}$$

taking the Fokker-Planck equation to determine P. Instead, it is better to argue directly that

$$\langle \varrho(x) \rangle = \left\langle \int_0^x \frac{\mathrm{d}x'}{\chi^2 \, u(x')} \right\rangle$$

$$= \int_0^x \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}u \, \frac{\exp\{-u^2/2\}}{\sqrt{2\pi}} \, \frac{1}{\chi^2(u)} \tag{3.2}$$

since the 1-point distribution $P_1(x: u)$ does not explicitly depend on x (u is stationary!). We have

$$U(x) = \mu \, x .$$

We would like to call μ the "drag" coefficient, defined by

$$\mu = \int_{-\infty}^{+\infty} \frac{\mathrm{d}u}{\sqrt{2\pi}} \, \exp\{-u^2/2\} \, \frac{1}{\chi^2(u)} . \tag{3.4}$$

This formula imposes a certain restriction on the, hitherto, rather arbitrary choice of χ^2: μ should be finite. For the case already mentioned:

$$\chi^2 = \varepsilon + u^2 \tag{3.5}$$

this is obvious and μ can be given explicitly [1]. Another interesting choice might be

$$\chi^2(u) = a \exp\{\tfrac{1}{2}\alpha u^2\} + b \exp\{\beta \, |u|\} \tag{3.6}$$

4. The Width of the Gaussian Estimate (the Dispersion)

We are now concerned with the parameter $E^2(x)$ defined by

$$E^2(x) = \langle \varrho^2(x) \rangle - \langle \varrho(x) \rangle^2$$
$$\equiv \langle \tilde{\varrho}(x)^2 \rangle, \tag{4.1}$$

where

$$\tilde{\varrho}(x) = \varrho(x) - \mu \tag{4.2}$$

is the process $\varrho(x)$ after a centering procedure. Again, there are two ways:

(a) by means of the Fokker-Planck equation (see the Appendix on page 1083 ff);
(b) by using the direct definition.

We use the second procedure, which was so successful in the former treatment [1] of the first-order moment (Section 3). Consider

$$\langle \varrho^2(x) \rangle = \left\langle \int_0^x \frac{\mathrm{d}x'}{\chi^2(u(x'))} \int_0^x \frac{\mathrm{d}x''}{\chi^2(u(x''))} \right\rangle$$

$$= \int_0^x \mathrm{d}x' \int_0^x \mathrm{d}x'' \iint_{-\infty}^{+\infty} \frac{\exp\left\{-\dfrac{u_1^2 - 2\gamma u_1 u_2 + u_2^2}{2(1-\gamma^2)}\right\}}{2\pi \sqrt{1-\gamma^2}}$$

$$\cdot \frac{1}{\chi^2(u_1)} \frac{1}{\chi^2(u_2)} \mathrm{d}u_1 \, \mathrm{d}u_2, \tag{4.3}$$

where

$$\gamma = \exp\{-|x_1 - x_2|/\lambda\}. \tag{4.4}$$

Owing to the symmetry in $(x_1 - x_2)$, we are allowed to write

$$\langle \varrho^2(x) \rangle = 2 \int_0^x \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \, \sigma(x_1 - x_2), \tag{4.5}$$

where σ is the correlation function of $\chi^{-2}(u(x))$. If — following our previous reasoning — we expect $\sigma(x_1 - x_2) \sim 0$ if $|x_1 - x_2| > \lambda$, we may rewrite the above equation in the form $(x \gg \lambda)$

$$\langle \varrho^2(x) \rangle = 2 \int_0^x \mathrm{d}x_1 \int_0^{x_1 - \lambda} \mathrm{d}x_2 \, \mu^2 + 2 \int_\lambda^x \mathrm{d}x_1 \int_{x_1 - \lambda}^{x_1} \sigma(x_1 - x_2) \, \mathrm{d}x_2$$

$$\approx 2\mu^2 \int_\lambda^x \mathrm{d}x_1 (x_1 - \lambda) + 2 \int_\lambda^x \mathrm{d}x_1 \, \lambda \, \sigma(0) + \lambda^2 \sigma(0)$$

$$= 2\mu^2 \left(\frac{x^2}{2} - \frac{\lambda^2}{2} - \lambda x + \lambda^2\right) + 2\lambda \sigma(0)(x - \lambda) + \lambda^2 \sigma(0)$$

$$= (\mu x)^2 + \mu^2 \lambda^2 - (2\mu^2 \lambda) x + 2\lambda \sigma(0) x - \sigma(0) \lambda^2. \tag{4.6}$$

Hence we obtain for $x \to \infty$

$$\langle \varrho^2(x) \rangle \approx \langle \varrho(x) \rangle^2 + 2\lambda (\sigma(0) - \mu^2) x - \sigma(0) - \mu^2) \tag{4.7}$$

or, in the same approximation

$$\langle \tilde{\varrho}(x) \rangle \approx 2\lambda (\sigma(0) - \mu^2) x + \text{const.} \tag{4.8}$$

Hence we expect the Gaussian distribution to spread out with x and we may write more rigorously

$$\underset{x \to \infty}{\text{Asympt}} \langle \tilde{\varrho}^2(x) \rangle = x \, \Delta \tag{4.9}$$

and expect

$$\Delta \approx 2\lambda (\sigma(0) - \mu^2) \tag{4.10}$$

as an estimate for the "dispersion constant" Δ. It is obvious that the above approximation would not be improved for a rigorous formula for Δ. However, it ensures the asymptotic $(x \to \infty)$ behaviour of $\tilde{\varrho}(x)$ and hence could serve as a guide for treating the Fokker-Planck equation.

5. The Dispersion Constant Δ

Using the representaion (4.3) for the variance of $\varrho(x)$ and introducing the Fourier transform of $\chi^{-2}(u)$ by

$$g(\xi) = \int e^{i\xi u} \frac{\mathrm{d}u}{\chi^2(u)}, \tag{5.1}$$

by the invariance of the scalar product under unitary transformations we obtain first of all another representation for the correlation function

$$\sigma(x) = \iint \exp\{-\tfrac{1}{2}(\xi^2 + 2\gamma \xi \eta + \eta^2)\} g(\xi) g(\eta) \, \mathrm{d}\xi \, \mathrm{d}\eta$$

$$= \sum_{n=0}^{\infty} \frac{(-\gamma)^n}{n!} A_n^2 \tag{5.2}$$

with

$$A_n = \int \exp\{-\xi^2/2\} g(\xi) \xi^n \, \mathrm{d}\xi. \tag{5.3}$$

For symmetric $g(\xi)$ all A_n will vanish for odd n and we are left with

$$\sigma(x) = \sum_{\varkappa=0} \frac{\exp\{-2\varkappa|x|/\lambda\}}{(2\varkappa)!} A_{2\varkappa}^2. \qquad (5.4)$$

Integrating twice with respect to x we obtain

$$\langle \varrho^2(x) \rangle = \sum_{\varkappa=0} \frac{2A_{2\varkappa}^2}{(2\varkappa)!} \int_0^x dx' \int_0^x dx'' \exp\left\{-\frac{2\varkappa(x'-x'')}{\lambda}\right\} \qquad (5.5)$$

or, since $A_0 = \mu$

$$\langle \varrho^2(x) \rangle - \langle \varrho(x) \rangle^2 \qquad (5.6)$$

$$= \lambda \sum_{\varkappa=1}^{\infty} \frac{2A_\varkappa^2}{2\varkappa(2\varkappa)!} x - \lambda^2 \exp\{-|x|/\lambda\} \, \mathrm{const} + \ldots,$$

where the dots denote decaying terms proportional to at least $\lambda^2 \exp\{-2x/\lambda\}$. Thus we have for the dispersion constant

$$\Delta = \sum_{\varkappa=1}^{\infty} \lambda \frac{A_{2\varkappa}^2}{\varkappa(2\varkappa)!} \qquad (5.7)$$

or explicitly

$$\Delta = \lambda \iint \exp\{-\tfrac{1}{2}(\xi^2+\eta^2)\} \, g(\xi) \, g(\eta)$$

$$\cdot \sum_{\varkappa=1}^{\infty} \frac{\xi^{2\varkappa} \eta^{2\varkappa}}{(2\varkappa)!\,\varkappa} \, d\xi \, d\eta. \qquad (5.8)$$

The sum in the integrand can be related to the exponential integral. For this purpose we introduce the function

$$\psi(z) = \sum_{\varkappa=1} \frac{z^{2\varkappa}}{2\varkappa(2\varkappa)!}. \qquad (5.9)$$

Then, by differentiating and re-integration one obtains easily

$$\psi(z) = \int_1^z \frac{\mathrm{Cosh}\, z' - 1}{z'} \, dz' + \mathrm{const}, \qquad (5.10)$$

which establishes ψ as related to the exponential hypercosine. We end up with the representation

$$\Delta = 2\lambda \iint_0^{\infty} \exp\{-(\xi^2+\eta^2)/2\} \, g(\xi) \, g(\eta) \, \psi(\xi\,\eta) \, d\xi \, d\eta. \qquad (5.11)$$

Appendix (Frank Pohl)

Even if the final value of the process $u(x)$ is fixed at, say, w, one obtains instead of Eq. (1.3)

$$\varrho(x, w) = \int_0^x \frac{dx'}{\chi^2(u(x'))} \bigg|_{u(x)=w}. \qquad (A.1)$$

The argument for the central-limit-theorem would not be changed for large x, since the above condition exerts just a local influence. Hence the conditional probability

$$F(x, w: \varrho) = \mathrm{prob}(x: w, \varrho)/\mathrm{prob}(x: w) \qquad (A.2)$$

will obey — according to Eq. (1.6) — the following Fokker-Planck-equation (with $\beta = 1$):

$$F_x = F_{ww} - wF_w - \frac{F_\varrho}{\chi^2(w)}. \qquad (A.3)$$

For its cumulant function $\psi(x, w: \lambda)$

$$\psi(x, w: \lambda) = \ln \int_{-\infty}^{+\infty} d\varrho \, F(x, w: \varrho) \, e^{i\lambda\varrho}$$

one gets

$$\psi_x = \psi_{ww} - w \, \psi_w + \frac{i\lambda}{\chi^2} + \psi_w^2. \qquad (A.4)$$

For large x one would expect a Gaussian for F as a function of ϱ and hence try the ansatz

$$\psi = C(\lambda) \, x + S(\lambda, w) \qquad (A.5)$$

which leads to the ordinary differential equation

$$C(\lambda) = S_{ww} - w S_w + \frac{i\lambda}{\chi^2} + S_w^2. \qquad (A.6)$$

Multiplying by $\exp(-w^2/2)/\sqrt{2\pi}$ one obtains

$$C(\lambda) = i\lambda \int \frac{e^{-w^2/2}}{\sqrt{2\pi}} \frac{dw}{\chi^2(w)} + \int \frac{e^{-w^2/2}}{\sqrt{2\pi}} S_w^2 \, dw \qquad (A.7)$$

which shows the dependence of $C(\lambda)$. Expanding C and S_w in terms of λ

$$C = i\lambda C_1 - \frac{\lambda^2}{2} C_2 - \ldots,$$

$$S_w = i\lambda S_{1,w} - \frac{\lambda^2}{2} S_{2,w} - \ldots$$

yields

$$C_1 = \int_{-\infty}^{+\infty} \frac{e^{-w^2/2}}{\sqrt{2\pi}} \frac{dw}{\chi^2(w)}, \qquad (A.8)$$

$$C_2 = \int_{-\infty}^{+\infty} \frac{e^{-w^2/2}}{\sqrt{2\pi}} 2 S_{1,w}^2 \, dw, \qquad (A.9)$$

where $S_{1,w}$ is the first term of the expansion, being obtained by solving the equation for the first cumulant; see [2], Equation (4.3a).

The initial conditions are not expected to have great influence. However, it is possible to deal rigorously with them; see [2].

It is evident that C_1 coincides with μ; furthermore it was numerically checked, that C_2 also coincides with Δ.

[1] P. Gräff and D. Pfirsch, Z. Naturforsch. **37a,** 804 (1982).
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[4] A. D. Wentzell, Theorie zufälliger Prozesse. Birkhäuser, Basel 1979.